Zeckendorf arithmetic for Lucas numbers

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Abstract

In this article we will be dedicated some algorithms of addition, subtraction, multiplication and division of two positive integers using Zeckendorf form. Such results find application in coding theory.

1 Introduction

There are few previous work in this area. Graham, Knuth and Patashnik [6] discussing the addition of 1 in the Zeckendorf representation, but have not talked about the actual arithmetic. Fliponi [4] did for addition and multiplication, and Freitag philips [2] for the subtraction and division [3]. Thus, no previous work has discussed arithmetic as a coherent whole, covering all major operations, including multiplication and division. All these algorithms have been implemented and tested on a computer. Most algorithms are developed by analogy with conventional arithmetic methods. For example, multiplication is carried out by adding appropriate multiples of the multiplicand, depending on the selected bit pattern of the multiplier. The division will use a sequence of test subtraction, as in the normal long division.

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2 Zeckendorf theorems for Lucas numbers

Lucas numbers are defined by the recursion formula:

$$\begin{cases} L_n = L_{n-1} + L_{n-2}, n \geq 2 \\ L_0 = 2, L_1 = 1. \end{cases}$$

and for all $n \geq 0$, we have the well-known $L_n = F_{n+1} + F_{n-1}$ where F_n is the nth Fibonacci number.

2.1 Zeckendorf decompositon method

To decompose an integer x of the form Zeckendorf $x = \sum_{r=0}^{\propto} \alpha_r L_r$ just follow the following steps:
1.Find the greater Lucas number $L_r \leqslant x$.
2.Do subtraction $X = x - L_r$, assign a 1 to e_r and keep this coefficient.
3.Assign X to x and repeat steps 1 et 2 until X have a zero.
4.Assign of 0 to e_i where $0 \leqslant i \leqslant r$ and $e_i \neq 1$.
The result of this decomposition is a vector of r elements that contains the coefficients e_i decomposition. Example decomposition of 50, this table shows the performance:

Lucas sequence	2	1	3	4	7	11	18	29	47	76
Vector e_r	0	0	1	0	0	0	0	0	1	0

So $50_L = 001000001$

Theorem 1. *Let n be a positive integer satisfying $0 \leqslant n \leqslant L_k$ for $k \geqslant 1$. Then*
$n = \sum_{i=0}^{k-1} \alpha_i L_i$ *where $\alpha_i \in \{0, 1\}$ such that*

$$\begin{cases} i)\alpha_i \alpha_{i+1} = 0, for(i \geqslant 0) \\ ii)\alpha_0 \alpha_2 = 0. \end{cases}$$

this representation is unique.

Proof: [1]

3 Addition

We take two positive integers a and b written in the form of Zeckendorf, obtainable form of a + b Zeckendorf repeating adding, at the same time, numbers of Lucas occupant in one of two numbers , say b , to another number a. This gives an initial amount for which figures are $d_i \in \{0, 1\}$, where each d_i is L_i its number of Lucas. For $d_i = 2$ does not exist because $n = 1 \rightarrow 2L_1 = L_0$, we replace 020 by 001 and $n \geq 2 \rightarrow 2L_n = L_{n+1} + L_{n-2}$, we replace 00200 by 01001. In way is equivalent model x 2 y z figures transforms to
(1 + x)0y (1 + z). This rule does not apply to terms with a weight of 1, which is covered by the special case below. If the combination 011 exists in the vector e_r, we will substitute it by 001. This step must be performed by scanning left to right through the performance. Here is a table that summarizes all possible cases of the addition in the representation of Zeckendorf:

Addition	Lucas weight	L_{i+1}	L_i	L_{i-1}	L_{i-2}
Consecutive 1		x	y	1	1
	becomes	x	$y+1$	0	0
Eliminate a 2	here $x \geq 2$	w	x	y	z
	becomes	$w+1$	$x-2$	y	$z+1$
Add,right bits		L_2	L_1	L_0	
$d_2 \geq 2$	here $x \geq 2$		x	0	
	becomes		0	1	
$d_2 \geq 2$			0	x	
	becomes	1	1	0	

Table 1: Adjustments and corrections in addition

This table shows the two additions examples 33+19 and 12+19 in zeckendorf representation:

a		1	0	0	0	1	0	0	0	$= 33$
b			1	0	0	0	0	1	0	$= 19$
initial sum		1	1	0	0	1	0	1	0	$= 52$
consecutive 1	1	0	0	0	0	1	0	1	0	$= 52$
becomes	1	0	0	0	0	1	0	1	0	$= 52$
check $33 + 19 = 52$										

Table 2: Example of addition $(33 + 19)$

a			1	0	0	0	1	0		$= 12$
b		1	0	0	0	0	1	0		$= 19$
initial sum		1	1	0	0	0	2	0		$= 31$
		1	1	0	0	0	0	1		$= 31$
consecutives 1		1	1	0	0	0	0	1		$= 31$
	1	0	0	0	0	0	0	1		$= 31$
becomes	1	0	0	0	0	0	0	1		$= 31$
check $12 + 19 = 31$										

Table 3: Example of addition $(12 + 19)$

4 Soustraction

For subtraction, $X - Y = Z$, where $X > Y$ and Z the difference. We start by subtracting all figures $x_i - y_i = z_i$, where $z_i \in \{0, 1. -1\}$ Values 0 and 1 have no problem, as they are valid representation in Zeckendorf. Where $z_i = -1$ is the most difficult. If in this case, go to the next bit 1 and is written in the Fibonacci rule $100 \rightarrow 011$ and write bit 1 rightmost pairs 1 is repeated until the bit 1 bit coincides with the -1 in the same position and eliminates replacing the box by 0 then 1 consecutive passes.

Soustraction	Lucas weights	L_{i+2}	L_{i+1}	L_i	L_{i-1}	L_{i-2}
eliminate -1		1	0	0	0	-1
		0	1	1	0	-1
	becomes	0	1	0	1	0

Table 4: Adjustments and corrections in subtraction

This table shows the example 42-32 in Zeckendorf representation.As elsewhere,different rewriting rules can be written in any order.

a	1	0	1	0	0	0	0	1	= 42
b	1	0	0	0	0	1	0	0	= 32
subtract bit by bit			1	0	0	-1	0	1	= 10
rewrite 1000				1	1	-1	0	1	= 10
rewrite 0110, cancelling -1				1	0	0	1	1	= 10
consecutive 1				1	0	1	0	0	= 10
becomes				1	0	1	0	0	= 10

Table 5: Example of subtraction $(42-32)$

5 Multiplication

Using the following results (propositions 1,2,3,4) and section 3 above, one can derive a multiplication method of integers in Zackendorf representaion.

Proposition 1. *If $n \geq 3$, then*

$$L_k L_{k+n} = \begin{cases} F_{n-1} + F_{n+1} + F_{2k+n\pm 1} & (k \;\; even), \\ F_{n-2} + F_{n+1} + F_{2k+n+1} + \sum_{j=1}^{k-2} F_{2j+n+2} & (k \geq 3, odd). \end{cases}$$

Proposition 2. *If $n \geq 5$, then*

$$2L_k L_{k+n} = \begin{cases} F_{n\pm 3} + F_{2k+n\pm 3} & (k \geq 4, even), \\ F_{n-4} + F_{2k+n+3} + \sum_{j=1}^{3} F_{2j+n-3} + \sum_{j=1}^{k-4} F_{2j+n+4} & (k \geq 5, odd). \end{cases}$$

Proposition 3. *If $n \geq 5$, then*

$$3L_k L_{k+n} = \begin{cases} \sum_{j=1}^{4} (F_{2j+n-5} + F_{2j+2k+n-5}) & (k \geq 4, even), \\ F_{n-4} + F_{n+3} + \sum_{j=1}^{3} F_{2j+2k+n-3} + \sum_{j=1}^{k-4} F_{2j+n+4} & (k \geq 5, odd). \end{cases}$$

Proposition 4. *If $n \geq 6$, then*

$$4L_k L_{k+n} = \begin{cases} \sum_{j=1}^{4} (F_{3j+n-8} + F_{3j+2k+n-8}) & (k \geq 6, even), \\ F_{n-4} + F_{n-2} + F_{n+1} + \sum_{j=1}^{3} F_{3j+2k+n-5} + \sum_{j=1}^{k-5} F_{2j+n+4} & (k \geq 5, odd). \end{cases}$$

Proofs: [5]

This example shows how to compute 17×10 in Zeckendorf representation:

a						1	0	1	0	0	1	=
b							1	0	1	0	0	=
Multiple of Luca 17												
multiple L_1						1	0	1	0	0	1	=
multiple L_2			1	0	0	0	0	1	0	0	0	=
multiple L_3			1	0	1	0	0	0	1	0	0	=
multiple L_4		1	0	1	0	1	0	0	1	0	0	=
multiple L_5	1	0	1	0	0	1	0	1	0	0	1	=
Accumulate appropriate multiples												
Add multiple of L_2			1	0	0	0	0	1	0	0	0	=
Add multiple of L_4		1	0	1	0	1	0	0	1	0	0	=
$L_2 + L_4$=		1	1	1	0	1	0	1	1	0	0	=
Eliminate 1 consecutive	1	0	0	1	0	1	1	0	0	0	0	=
Eliminate 1 consecutive	1	0	0	1	1	0	0	0	0	0	0	=
becomes=	1	0	1	0	0	0	0	0	0	0	0	=

Table 6: Example of Zeckendorf multiplication (17×10)

6 Division

Using the following proposition 5 and section 4 above, one can derive a division method of integers in Zackendorf representation.

Proposition 5. *First,for $k = 4m$ and n odd, we obtain*

$$\frac{F_{kn}}{F_n} = \sum_{r=1}^{m} \left(L_{(k-4r+3)n} + L_{(k-4r+1)n} \right),$$

and thus

$$\frac{F_{kn}}{F_n} = S_{k,n},$$

say, where

$$S_{k,n} = \sum_{r=0}^{[k/4]-1} \left(F_{(k-4r-1)n+1} + \left(\sum_{s=1}^{n-2} F_{(k-4r-1)n-2s} \right) + F_{(k-4r-3)n+1} + F_{(k-4r-3)n-2} \right).$$

We similarly work through the other cases, where n is odd and $k \equiv 1, 2$ and $3 \mod 4$. In each case, the "most significant" part of the Zeckendorf form is $S_{k,n}$. The precise Zackendorf form is

$$\frac{F_{kn}}{F_n} = S_{k,n} + e_{k,n},$$

where the least significant part of the Zeckendorf sum is

$$e_{k,n} = \begin{cases} 0, & k \equiv 0 \mod 4), \\ F_2, & k \equiv 1 \mod 4), \\ F_{n+1} + F_{n-1}, & k \equiv 2 \mod 4), \\ F_{2n+1} + \sum_{r=1}^{n-1} F_{2n-2r}, & k \equiv 3 \mod 4). \end{cases}$$

Proof:[3]

This example shows how to compute $250 \div 17$ in Zeckendorf representation:

a		1	0	1	0	1	0	0	0	1	0	0	=
b							1	0	1	0	0	1	=
Make lucas Multiples of divisor													
multiple L_1							1	0	1	0	0	1	=
multiple L_2				1	0	0	0	0	1	0	0	0	=
multiple L_3				1	0	1	0	0	0	1	0	0	=
multiple L_4			1	0	1	0	1	0	0	1	0	0	=
multiple L_5		1	0	1	0	0	1	0	1	0	0	1	=
multiple L_6	1	0	1	0	1	0	0	0	0	0	0	1	=
Trial subtraction													
$L5$ residue=				1	0	0	1	0	1	0	1	0	=
$L2$ residue=							1	0	0	0	1	0	=
quotient=							1	0	1	0	0	1	=
remainder =							1	0	0	0	1	0	=

Table 7: Example of Zeckendorf division $(250 \div 17)$

7 Conclusion

Although we have highlighted the main arithmetic operations on integers Zeckendorf, this arithmetic should not stay more than a curiosity. In future research, we plan to study the applications of our results to other areas of mathematics such as error correcting codes.

References

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